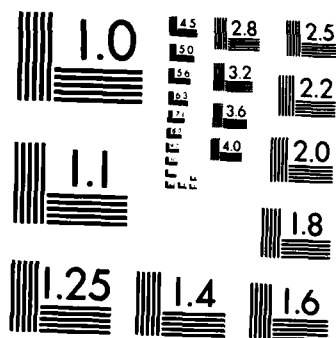


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AN EXTENSION OF THE KREISS MATRIX  
THEOREM

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**AN EXTENSION OF THE KREISS MATRIX THEOREM**

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**ABSTRACT**

A new condition is shown to be equivalent to the other conditions of the Kreiss Matrix Theorem for power bounded families of matrices. This new condition is important for the application of the theory of pseudo-difference operators to stability estimates for variable coefficient finite difference equations. As an example of the usefulness of this new condition, we use it to prove stability of the leap frog scheme for hyperbolic equations with variable coefficients.

AMS (MOS) Subject Classifications: 15A60, 65M15

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### SIGNIFICANCE AND EXPLANATION

The existence of a symmetrizer is a vital part of the new stability estimate obtained in [10] for general variable coefficient linear finite difference equations. It allows a Gårding inequality for pseudo-difference operators to be utilized and is the finite difference analogue of the symmetrizer for partial differential equations. The theorem which appears in this paper links the existing necessary and sufficient conditions for the power boundedness of a family of matrices to the existence of a matrix symmetrizer. An example is given to indicate how the new condition allows stability estimates to be obtained for finite difference schemes which do not satisfy the hypotheses of previous stability theorems.

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# AN EXTENSION OF THE KREISS MATRIX THEOREM

Bruce A. Wade

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The Kreiss Matrix Theorem [2] gives necessary and sufficient conditions for the power boundedness of a family of matrices. The power boundedness property is related to the stability of variable coefficient finite difference equations which, in turn, is related to convergence. In order to obtain stability and convergence estimates for variable coefficient finite difference equations it is useful to introduce a new condition in the Kreiss Matrix Theorem. This condition involves the construction of a matrix called a symmetrizer.

The symmetrizer is used in [10] to obtain stability estimates for variable coefficient finite difference equations through the use of a Gårding inequality and the theory of pseudo-difference operators. A similar theory of pseudo-difference operators for regular elliptic systems of difference equations was developed by Bube and Strikwerda in [1]. Michelson [6] has also developed a theory of pseudo-difference operators. The matrix constructed is called a symmetrizer because it is the finite difference analogue of the symmetrizer for the Cauchy problem in the theory of partial differential equations. We include an example of the application of the new symmetrizer condition to the proof of the stability of the leap frog difference method for variable coefficient problems.

Tadmor [9] generalized the Lax-Wendroff [5] sufficient condition for stability. The Lax Wendroff version of the theorem involves the numerical radius of the amplification

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matrix while Tadmor's condition uses a generalized numerical radius. The generalized numerical radius condition was shown to be necessary and sufficient for the power boundedness of a family of matrices. We show how the new symmetrizer condition is related to Tadmor's condition. In the interest of showing this relationship, we include more implications in the proof than are necessary to establish the theorem.

Consider the vector space  $\mathbb{C}^m$  and let  $A$  and  $\Omega$  be  $m$  by  $m$  complex matrices with  $\Omega$  Hermitian and positive definite. Define, for  $x, y \in \mathbb{C}^m$ ,

$$\langle x, y \rangle := x \cdot \bar{y}, \quad \|x\|^2 := \langle x, x \rangle.$$

and

$$\|A\| := \sup_{x \neq 0} \frac{\|Ax\|}{\|x\|}.$$

Define the  $\Omega$ -numerical radius of  $A$  to be

$$r_{\Omega}(A) := \sup_{x \neq 0} \frac{|\langle \Omega Ax, x \rangle|}{|\langle \Omega x, x \rangle|}.$$

The real part of a matrix  $A$ , namely  $\frac{1}{2}(A^* + A)$ , where  $*$  denotes the conjugate transpose, will be denoted by  $\Re A$ . Given two Hermitian matrices  $A$  and  $B$ , one says that  $A \leq B$  if, for each  $x \in \mathbb{C}^m$ ,

$$\langle Ax, x \rangle \leq \langle Bx, x \rangle.$$

Given an infinite family of  $m$  by  $m$  complex matrices denoted by  $\mathcal{F}$ , the following theorem contains four known equivalent conditions as well as the new symmetrizer condition. The original version was given by Kreiss in [2].

### Theorem (Kreiss Matrix Theorem)

The following are equivalent:

A) There exists  $C_A > 0$  such that for all  $A \in \mathcal{F}$  and for each  $n \in \mathbb{Z}_+$ ,  $\|A^n\| \leq C_A$ .

R) There exists  $C_R > 0$  such that for all  $A \in \mathcal{F}$  and for all  $|z| > 1$ ,

$$\|(zI - A)^{-1}\| \leq C_R(|z| - 1)^{-1}.$$

H) There exists  $C_H > 0$  such that for each  $A \in \mathcal{F}$  there is a positive definite, Hermitian matrix  $H$  which satisfies  $C_H^{-1}I \leq H \leq C_H I$  and  $A^* H A \leq H$ .

N) There exist  $C_N > 0$ ,  $C_0 > 0$  such that for each  $A \in \mathcal{F}$  there is a positive definite, Hermitian matrix  $N$ , called a symmetrizer, which satisfies  $C_N^{-1}I \leq N \leq C_N I$  and, for each  $|z| \geq 1$ ,

$$\Re(N(I - z^{-1}A)) \geq C_0(1 - |z|^{-1})I.$$

$\Omega$ ) There exists  $C_\Omega > 0$  such that for each  $A \in \mathcal{F}$  there exists a positive definite, Hermitian matrix  $\Omega$  which satisfies  $C_\Omega^{-1}I \leq \Omega \leq C_\Omega I$  and, for all  $n \in \mathbb{Z}_+$ ,  $r_\Omega(A^n) \leq 1$ .

The original paper of Kreiss [2] included the conditions A, R, H and another condition which is commonly called S. We omit the condition S for brevity. The condition  $\Omega$  was introduced by Tadmor in [9]. The condition N, which will be called the symmetrizer condition, is new. Although the symmetrizer condition may appear to be weaker than Tadmor's generalized numerical radius condition, we show directly that these conditions are equivalent, thus elucidating the relationship between them.



# PROOF:

The proof that condition A implies the resolvent condition R and that R implies H are contained in [2]. Kreiss proved that R implies H by first proving an intermediate condition, which he called S. This is the most difficult part of the theorem and involves estimates on the behavior of the eigenvalues, especially for those near the unit circle. Tadmor proved in [9] that  $\Omega$  implies A by utilizing the generalized Halmos inequality,  $r_{\Omega}(A^n) \leq (r_{\Omega}(A))^n$ , (see [8,9]), and the inequality  $\|A\| \leq 2r_I(A)$ . This last inequality can be proved by writing A as the sum of its Hermitian and skew parts and noting that the spectral radius of a normal matrix equals its spectral norm.

We now prove that the symmetrizer condition is equivalent to the others.

We begin by proving that condition H implies the symmetrizer condition. N. Assuming H holds, let  $A \in \mathcal{F}$  and let H be the corresponding positive definite Hermitian matrix. Expanding the relation

$$0 \leq (I - z^{-1}A)^* H (I - z^{-1}A),$$

we have

$$\begin{aligned} 0 &\leq H - 2\Re(z^{-1}HA) + |z|^{-2}A^*HA \\ &\leq H - 2\Re(z^{-1}HA) + |z|^{-2}H \\ &= 2\Re(H(I - z^{-1}A)) + (|z|^{-2} - 1)H. \end{aligned}$$

Thus

$$\begin{aligned} \frac{1}{2}(1 - |z|^{-1})H &\leq \frac{1}{2}(1 - |z|^{-2})H \\ &\leq \Re(H(I - z^{-1}A)), \end{aligned}$$

and the symmetrizer condition follows with N taken as H.

The symmetrizer condition, N, implies  $\Omega$ . To see this, note that if N holds then

$$\Re(N(I - z^{-1}A)) \geq C_0(1 - |z|^{-1})I$$

$$\geq 0.$$

This implies that

$$\Re(N) \geq \Re(Nz^{-1}A), \quad \text{for all } |z| \geq 1.$$

Given  $x \in C^m$ , choose  $z$  such that

$$\Re(\langle NAx, x \rangle z^{-1}) = |\langle NAx, x \rangle|.$$

Therefore,

$$|\langle NAx, x \rangle| \leq \langle Nx, x \rangle, \quad \text{for all } x \in C^m,$$

which implies  $r_N(A) \leq 1$ . This, together with the generalized Halmos inequality [8,9], proves that condition  $\Omega$  holds.

Conversely, assume the condition  $\Omega$  holds. We will show that this implies the symmetrizer condition, N. This part of the proof is included only to show the relation between the symmetrizer condition and Tadmor's generalized numerical radius condition. We have, for  $x \in C^m$ ,

$$\begin{aligned} \Re(\langle \Omega(I - z^{-1}A)x, x \rangle) &= \Re(\langle \Omega x, x \rangle) - \Re(\langle z^{-1}\Omega Ax, x \rangle) \\ &\geq \langle \Omega x, x \rangle - |z|^{-1}|\langle \Omega Ax, x \rangle| \\ &\geq \langle \Omega x, x \rangle - |z|^{-1}\langle \Omega x, x \rangle \\ &= (1 - |z|^{-1})\langle \Omega x, x \rangle \\ &\geq C_\Omega^{-1}(1 - |z|^{-1})\langle x, x \rangle. \end{aligned}$$

This implies the symmetrizer condition with  $N := \Omega$ , and  $C_N := C_\Omega$ , and  $C_0 := C_\Omega^{-1}$ .

The symmetrizer condition is also closely related to the resolvent condition. If condition N holds, then

$$\begin{aligned} C_0(1 - |z|^{-1}) &\leq \|\Re(N(I - z^{-1}A))\|, \\ &\leq \|N\|\|I - z^{-1}A\|, \\ &\leq C_N\|I - z^{-1}A\|, \end{aligned}$$

where the last inequality follows from observing  $|N| = \omega_I(N) \leq C_N$ , and the first follows from the Cauchy-Schwarz inequality. This is easily seen to be equivalent to the condition R. The proof of the theorem is complete.

We now present a simple example which illustrates the use of the symmetrizer condition in proving the stability of variable coefficient finite difference equations. The methods of Kreiss [3], Parlett [7] and Lax and Nirenberg [4] do not apply to the difference scheme given here; however, the symmetrizer condition can be used to obtain the stability estimate. We make use of the calculus of pseudo-difference operators in the form developed by Bube and Strikwerda [1] and Michelson [6]. A complete presentation of the application of the symmetrizer condition to stability and convergence estimates of general finite difference equations along with applications will be presented in [10].

Consider the variable coefficient initial value problem

$$u_t = a(x, t)u_x, \quad t \geq 0, \quad x \in \mathbf{R}$$

$$u(x, 0) = u_0(x),$$

where  $u_0(\cdot) \in L^2(\mathbf{R})$  is a scalar valued function and  $a(x, t)$  is constant outside of a compact set. We approximate the partial differential equation by the leap frog difference scheme

$$v_m^{n+1} - v_m^{n-1} = \lambda a_{m,n} (v_{m+1}^n - v_{m-1}^n), \quad n \geq 1, \quad m \in \mathbf{Z}$$

$$v_m^0 := u_0(mh) \tag{1}$$

$$v_m^1 := v_m^0 + \frac{1}{2} \lambda a_{m,0} (v_{m+1}^0 - v_{m-1}^0),$$

where  $a_{m,n} := a(mh, nk)$  and  $\lambda := k/h$  is constant. We assume that  $\sup_{x,t} \lambda |a(x, t)| < 1$ .

Define the discrete function

$$\Delta_m^n := \begin{cases} (1/2k)v_m^0, & \text{if } n = -1; \\ (1/2k) \left( v_m^0 - \frac{1}{2} \lambda a_{m,0} (v_{m+1}^0 - v_{m-1}^0) \right), & \text{if } n = 0; \\ 0, & \text{else,} \end{cases}$$

then it is easy to check that the difference equation (1) is equivalent to

$$v_m^{n+1} - v_m^{n-1} = \lambda a_{m,n} (v_{m+1}^n - v_{m-1}^n) + 2k \Delta_m^n, \quad n, m \in \mathbf{Z}, \quad (2)$$

where we now consider the difference equation as being for all time levels and take  $v^n$  to be 0 if  $n$  is negative. Note that  $\Delta_m^n$  essentially acts as a "delta function" and its nonzero elements are of order  $1/k$ .

We now reduce this difference scheme to a two level scheme. If we define

$$W_m^n := \begin{pmatrix} v_m^{n+1} \\ v_m^n \end{pmatrix},$$

and

$$F_m^n := \begin{pmatrix} 2\Delta_m^n \\ 0 \end{pmatrix},$$

equation (2) may be written as

$$W_m^n = G(m, n) W_m^{n-1} + k F_m^n, \quad n, m \in \mathbf{Z} \quad (3)$$

where

$$G(m, n) := \begin{pmatrix} \lambda a_{m,n} (T - T^{-1}) & 1 \\ 1 & 0 \end{pmatrix},$$

and  $T$  is the translation operator in  $x$ .

The symbol of the difference operator,  $G$ , is defined to be

$$\hat{G}(m, n, \xi) := \begin{pmatrix} 2\lambda i a_{m,n} \sin \xi h & 1 \\ 1 & 0 \end{pmatrix}, \quad |\xi| \leq \frac{\pi}{h}.$$

We note that for each value of  $(m, n)$  the eigenvalues of  $\hat{G}(m, n, \xi)$  are on the unit circle, in particular the scheme is nondissipative. Therefore, the method of Kreiss [3] and Parlett [7] for proving stability does not apply. The sharp Gårding inequality of Lax and Nirenberg

[4] also does not apply since the norm of the symbol is not bounded by 1. We will utilize the symmetrizer condition in order to prove that this difference scheme is stable.

The theory of pseudo-difference operators is employed in the context of the discrete Sobolev space

$$H_{\eta,h} := \{v : h\mathbf{Z} \times k\mathbf{Z} \rightarrow \mathbf{R} : \quad \hat{v}(\xi, s) \in L^2(\mathbf{D}_{\eta,h})\},$$

where  $\mathbf{D}_{\eta,h} := \{(\xi, \tau) : |\xi| \leq \frac{\pi}{h}, |\tau| \leq \frac{\pi}{k}\}$ ,  $s = \eta + i\tau$ , with  $\eta \geq \eta_0 > 0$ , and  $\hat{v}(\xi, s)$  denotes the discrete Fourier transform in space and Laplace transform in time of  $v_m^n$  defined by

$$\hat{v}(\xi, s) = \frac{hk}{2\pi} \sum_{n,m=-\infty}^{\infty} e^{-snk - i\xi mh} v_m^n.$$

The corresponding norm is

$$\|v\|_{\eta,h}^2 = hk \sum_{m,n=-\infty}^{\infty} |e^{-2\eta k} v_m^n|^2.$$

See Michelson [6] or Wade [10] for details on the pseudo-difference operator theory as well as the discrete transform and discrete Sobolev space.

Consider the family of matrices

$$\mathcal{F} := \{e^{-\eta_0 k} \hat{G}(m, n, \xi) : m, n \in \mathbf{Z}, |\xi| \leq \frac{\pi}{h}\},$$

where  $\eta_0$  is any positive number. Since the eigenvalues of every element of  $\mathcal{F}$  are distinct, inside the unit circle and uniformly separated, it is clear that the resolvent equation holds. Therefore, there is a constant  $C_0 > 0$  such that for each element of  $\mathcal{F}$  there is a symmetrizer,  $\hat{N}(m, n, \xi)$ , which satisfies

$$\Re \left( \hat{N}(m, n, \xi) (I - e^{-sk} \hat{G}(m, n, \xi)) \right) \geq C_0 (1 - e^{-\eta_0 k}) I. \quad \eta \geq \eta_0 > 0. \quad (4)$$

We now apply a Gårding inequality for pseudo-difference operators. (see [1], [6], or [10]). The Gårding inequality essentially says that the condition (4), which holds for the symbol of the difference operator, is also true for the variable coefficient operator modulo lower order terms. We have

$$C_0 \left( \frac{1 - e^{-\eta k}}{k} \right) \|W\|_{\eta, h}^2 \leq \Re k^{-1} (W, N(I - G(n, m)S^{-1})W)_{\eta, h} + c \|W\|_{\eta, h}^2 \quad (5)$$

where  $S$  is the shift operator in  $t$ .

By choosing  $\eta_0$  larger, and the constant  $C_0$  smaller, and using that  $W$  satisfies the equation (3), this implies

$$C_0 \left( \frac{1 - e^{-\eta k}}{k} \right) \|W\|_{\eta, h}^2 \leq \Re (W, NF)_{\eta, h} \leq C \|W^0\|_h^2, \quad \eta \geq \eta_0.$$

The last inequality follows from the fact that the grid function  $F$  is supported on only two time levels. In terms of the original variable  $v_m^n$  this is

$$\left( \frac{1 - e^{-\eta k}}{k} \right) \|v\|_{\eta, h}^2 \leq Ch \sum_{m \in \mathbb{Z}} |v_m^0|^2, \quad \eta \geq \eta_0 > 0, \quad (6)$$

for some constant  $C$ .

The estimate (6) is equivalent to the standard notion of stability. By 'standard notion of stability' we mean the definition which requires the space norm of the solution at any time level  $n$  to be bounded by a constant depending on  $nk$  times the space norm of the initial data of the difference scheme. The reader is referred to the paper of Bube and Strikwerda [1] for a similar estimate which uses a Gårding's inequality for elliptic systems.

Finally we comment on the reason that the symmetrizer condition allows estimates for more general difference schemes. The previous methods for proving stability involved

the conditions  $H$  or  $\Omega$ . These conditions involve nonlinear operations on the elements of the family of matrices, while condition  $N$  is essentially a linear operation on the elements of the family. The condition  $H$  contains the term  $A^*HA$  and the condition  $\Omega$  contains the absolute value of  $\langle \Omega, \cdot \rangle$ , both of which are nonlinear operations on the matrices in the family. The symmetrizer condition utilizes only the multiplication of elements of the family by a matrix, in addition to the operation of taking the real part of a matrix. The use of only linear relations allows for greater flexibility in the application of Gårding inequalities.

Estimates for more general schemes as well as other results are contained in [10].

## References

- [1] K. P. Bube and J. C. Strikwerda, "Interior regularity estimates for elliptic systems of difference equations," *SIAM J. Num. Anal.* **4**, 1983, pp. 653-670.
- [2] H.-O. Kreiss, "Über die Stabilitätsdefinition für Differenzengleichungen die Differentialgleichungen approximieren," *Nordisk. Tidskr. Inf. behandling (BIT)* **2**, 1962, pp. 153-181.
- [3] H.-O. Kreiss, "On difference approximations of the dissipative type for hyperbolic differential equations," *Comm. Pure Appl. Math.* **14**, 1964, pp. 335-353.
- [4] P. D. Lax and L. Nirenberg, "On the stability of difference schemes: a sharp form of Gårding's inequality," *Comm. Pure Appl. Math.* **19**, 1966, pp. 473-492.
- [5] P. D. Lax and B. Wendroff, "Difference schemes for hyperbolic equations with high order of accuracy," *Comm. Pure Appl. Math.* **18**, 1964, pp. 384-398.
- [6] D. Michelson, "Stability theory of difference approximations for multidimensional initial-boundary value problems," *Math. Comp.* **40**, 1983, pp. 1-45.
- [7] B. N. Parlett, "Accuracy and dissipation in difference schemes," *Comm. Pure Appl. Math.* **19**, 1966, pp. 111-123.
- [8] C. Pearcy, "An elementary proof of the power inequality for the numerical radius," *Michigan Math. J.* **13**, 1966, pp. 289-291.



- [9] E. Tadmor, "The equivalence of  $L_2$ -stability, the resolvent condition, and strict  $H$ -stability," *Lin. Alg. and Appl.* **41**, 1981, pp. 151-159.
- [10] B. Wade, Ph. D. Thesis, in preparation.

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